PERGAMON

Applied Mathematics Letters 14 (2001) 189–194

Applied Mathematics Letters

www.elsevier.nl/locate/aml

A Note on Singular Nonlinear Boundary Value Problems for the One-Dimensional p-Laplacian

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(Received January 2000; accepted March 2000)

Communicated by R. P. Agarwal

Abstract—Existence theorems about the positive solution for the singular equation $(\varphi_p(y'))' + f(t,y) = 0$, $y(0) = y(1) = 0$ are established. The results are obtained by using a fixed-point theorem in cones.

Keywords—One-dimensional singular p-Laplace equation, Cone, Compact map, Fixed point.

1. INTRODUCTION

The boundary value problem for the one-dimensional p-Laplacian

$$\begin{gathered}(\varphi_p(y'))' + f(t,y) = 0, \qquad t \in (0,1),\\ y(0) = y(1) = 0,\end{gathered} \tag{1}$$

where $\varphi_p(s) = |s|^{p-2}s, p > 1$, has been studied extensively. For details, see, for example, [1–4]. The boundary value problem treated in the above-mentioned references is not able to possess singularity.

In [5], Wang and Gao considered the BVP

$$\begin{gathered}(\varphi_p(y'))' + k(t)f(y) = 0, \qquad t \in (0,1),\\ y(0) = y(1) = 0.\end{gathered}$$

In the article, they supposed that $f(u)$ is nonincreasing in $(0,+\infty)$. It was a critical condition in the proof. On the other hand, many authors have considered a particular case of (1) for $p = 2$, see [6–8],

$$\begin{gathered}y'' + f(t,y) = 0, \qquad t \in (0,1),\\ y(0) = y(1) = 0.\end{gathered}$$

They all assumed that f was monotone for y. Monotone condition is very important.

The authors are supported by the National Science Foundation of China (19971037). We thank D. O'Regan for his help.

0893-9659/00/$ - see front matter Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
PII: S0893-9659(00)00134-8

This paper discusses the singular boundary value problem (1), but we eliminate the monotone condition. We obtain an existence theorem by using a fixed-point theorem in cones which no one has used to study p-Laplacian.

The following well-known lemma is very crucial in our arguments, see [9, p. 240].

LEMMA 1.1. *Let X be a Banach space, $K \subseteq X$ a cone in X. For $0 < \delta < r$, define $K_r = \{y \in K : |y| < r\}$. Assume that $F : \overline{K_r} \to K$ is a compact map such that*

(a) $Fy \neq \lambda y$, $\forall\, |y| = r$, *and* $\lambda > 1$;
(b) $Fy \neq \lambda y$, $\forall\, |y| = \delta$, *and* $\lambda < 1$;
(c) $\inf\{|Fy| : |y| = \delta\} > 0$.

Then F has a fixed point in $\overline{K_r} \backslash K_\delta$.

COROLLARY 1.1. *$F, \overline{K}_r, K$ are defined as Lemma 1.1, if*

(a) $|Fy| \leq |y| = r$, $\forall\, |y| = r$;
(b) $|Fy| \geq |y| = \delta$, $\forall\, |y| = \delta$.

Then F has a fixed point in $\overline{K_r} \backslash K_\delta$.

To be precise, we write $C^+[0,1] = C([0,1],[0,\infty))$. $B(\theta, \alpha) = \{y \in C^+[0,1] : \|y\|_0 < \alpha\}$. $B[0,1] = \{y \in C^+[0,1] : y(0) = y(1) = 0,\ y \text{ is convex}\}$. It is easily verified that $B[0,1] \subset C^+[0,1]$ is a cone. $B_\alpha = B(\theta,\alpha) \cap B[0,1]$. $\partial B_\alpha = \partial B(\theta, \alpha) \cap B[0,1]$.

2. MAIN RESULTS AND PROOF

THEOREM 2.1. *Suppose the following.*

(H1) $f \in C((0,1) \times [0,\infty),[0,\infty))$, $f(t,y) \leq p(t)q(y)$, *where* $q \in C([0,\infty),(0,\infty))$, $0 < \int_0^{1/2} \varphi_p^{-1}(\int_s^{1/2} p(r)\,dr)\,ds + \int_{1/2}^1 \varphi_p^{-1}(\int_{1/2}^s p(r)\,dr)\,ds < \infty$.

(H2) $\exists\, \alpha > 0$, *such that*

(i)

$$\max\left\{\int_0^{1/2} \varphi_p^{-1}\left(\int_s^{1/2} p(r) \max_{y\in[g_\alpha(r),\alpha]} q(y)\,dr\right) ds,\ \int_{1/2}^{1} \varphi_p^{-1}\left(\int_{1/2}^{s} p(r) \max_{y\in[g_\alpha(r),\alpha]} q(y)\,dr\right) ds\right\} \leq \alpha;$$

(ii) $f(t,0) \leq f(t,u)$, $0 < t < 1$, $0 \leq u \leq \alpha$. $\exists\, t_0 \in (0,1)$, *such that* $f(t_0, 0) > 0$, *where*

$$g_\alpha(t) = \begin{cases} \alpha t, & 0 \leq t \leq \dfrac{1}{2}, \\ \alpha(1-t), & \dfrac{1}{2} \leq t \leq 1. \end{cases}$$

(H3) $\exists\, 0 < \beta < \alpha$, *such that*

$$\min\left\{\int_0^{1/2} \varphi_p^{-1}\left(\int_s^{1/2} \min_{y\in[g_\beta(r),\beta]} f(r,y)\,dr\right) ds,\ \int_{1/2}^{1} \varphi_p^{-1}\left(\int_{1/2}^{s} \min_{y\in[g_\beta(r),\beta]} f(r,y)\,dr\right) ds\right\} \geq \beta.$$

Then BVP (1) has at least one solution $y \in C^1[0,1], \varphi_p(y') \in C(0,1)$, and $\beta \leq \|y\|_0 \leq \alpha$.

For $\forall\, y \in \overline{B(\theta,\alpha)}$, define

$$x(t) := \int_0^t \varphi_p^{-1}\left(\int_s^t f(r,y(r))\,dr\right) ds - \int_t^1 \varphi_p^{-1}\left(\int_t^s f(r,y(r))\,dr\right) ds, \qquad 0 < t < 1.$$

Clearly, $x(t)$ is continuous, nondecreasing in $(0,1)$ and $x(0+) < 0 < x(0-)$. Thus, $x(t)$ has zeros in $(0,1)$. Let A be a zero of $x(t)$ in $(0,1)$. Then,

$$\int_0^A \varphi_p^{-1}\left(\int_s^A f(r,y(r))\,dr\right) ds = \int_A^1 \varphi_p^{-1}\left(\int_A^s f(r,y(r))\,dr\right) ds. \tag{2}$$

Define the operator

$$T : \overline{B(\theta,\alpha)} \to B[0,1] \text{ by}$$

$$(Ty)(t) := \begin{cases} \displaystyle\int_0^t \varphi_p^{-1}\left(\int_s^A f(r,y(r))\,dr\right) ds, & 0 \le t \le A, \\ \displaystyle\int_t^1 \varphi_p^{-1}\left(\int_A^s f(r,y(r))\,dr\right) ds, & A \le t \le 1, \end{cases} \tag{3}$$

where A is satisfied (2).

LEMMA 2.1. *The operator T is compact from $\overline{B(\theta,\alpha)}$ to $B[0,1]$.*

PROOF. We first show $T\overline{B(\theta,\alpha)}$ is bounded. Put

$$q_\alpha = \sup\{q(y) : y \in [0,\alpha]\},$$

$$u_\alpha(t) := \begin{cases} \displaystyle\int_0^t \varphi_p^{-1}\left(\int_s^{A^*} p(r)\,dr\right) ds\varphi_p^{-1}(q_\alpha), & 0 \le t \le A^*, \\ \displaystyle\int_t^1 \varphi_p^{-1}\left(\int_{A^*}^s p(r)\,dr\right) ds\varphi_p^{-1}(q_\alpha), & A^* \le t \le 1, \end{cases}$$

where A^* is a zero of the function

$$x(t) := \int_0^t \varphi_p^{-1}\left(\int_s^t q(r)\,dr\right) ds - \int_t^1 \varphi_p^{-1}\left(\int_t^s q(r)\,dr\right) ds, \qquad 0 < t < 1,$$

and

$$\begin{aligned} u(t) &= (Ty)(t), & \forall y \in \overline{B(\theta,\alpha)}, \quad 0 < t < 1, \\ u_\theta(t) &= (T\theta)(t), & 0 < t < 1. \end{aligned}$$

So

$$f(t,0) \le f(t,y) \le q(t)q_\alpha, \tag{4}$$
$$(\varphi_p(u'_\alpha))' = -p(t)q_\alpha, \qquad t \in (0,1), \tag{5}$$
$$(\varphi_p(u'))' = -f(t,y), \qquad t \in (0,1), \tag{6}$$
$$(\varphi_p(u'_\theta))' = -f(t,0), \qquad t \in (0,1). \tag{7}$$

We can obtain that $u_\theta(t) \le u(t) \le u_\alpha(t)$, $t \in (0,1)$. Otherwise, there would exist $t_0 \in (0,1)$ at which $u(t_0) > u_\alpha(t_0)$, and hence, there would exist an interval (a,b) such that $u(t) > u_\alpha(t)$ in (a,b) and $u(a) - u_\alpha(a) = u(b) - u_\alpha(b) = 0$. Let $m = u(B) - u_\alpha(B)$ be the positive maximum of $u(t) - u_\alpha(t)$ on $[a,b]$. Then $B \in (a,b)$ and $u'(B) = u'_\alpha(B)$. Integrating both sides of the equalities (5) and (6) over $[s,B]$, $a < s < B$, we get

$$u'_\alpha(s) = \varphi_p^{-1}\left(\varphi_p(u'_\alpha(B)) + q_\alpha \int_s^B p(r)\,dr\right),$$

$$u'(s) = \varphi_p^{-1}\left(\varphi_p(u'(B)) + \int_s^B f(r,y(r))\,dr\right).$$

Integrating both sides of the above equality from a to B, we obtain

$$u_\alpha(B) - u_\alpha(a) = \int_a^B \varphi_p^{-1}\left(\varphi_p\left(u_\alpha'(B)\right) + q_\alpha \int_s^B p(r)\,dr\right) ds,$$

$$u(B) - u(a) = \int_a^B \varphi_p^{-1}\left(\varphi_p\left(u'(B)\right) + \int_s^B f(r, y(r))\,dr\right) ds.$$

Consequently, we are led to a contraction $0 < m = u(B) - u_\alpha(B) \le 0$ using inequality (4). It is similar that $u_\theta(t) \le u(t)$, $t \in (0,1)$. Hence,

$$\|Ty\|_0 \le \|u_\alpha\|_0 = u_\alpha\left(A^*\right), \qquad \forall y \in \overline{B(\theta,\alpha)}. \tag{8}$$

We next show the equicontinuity of $T\overline{B(\theta,\alpha)}$ on $[0,1]$. For any $\varepsilon > 0$, from the continuity of $u_\alpha(t)$ on $[0,1]$, it follows that there is a $\delta_1 \in (0, 1/4)$ such that

$$u_\alpha\left(2\delta_1\right), \qquad u_\alpha\left(1 - 2\delta_1\right) < \varepsilon.$$

If $(Ty)(A) < \varepsilon$, then for any $t_1, t_2 \in [0,1]$,

$$\left|(Ty)\left(t_1\right) - (Ty)\left(t_2\right)\right| \le |(Ty)(A) - (Ty)(0)| < \varepsilon.$$

If $(Ty)(A) > \varepsilon$, then $A \in [2\delta_1, 1 - 2\delta_1]$, and hence, for $t \in [\delta_1, 1 - \delta_1]$,

$$|(Ty)'(t)| = \left|\varphi_p^{-1}\left(\int_t^A f(r, y(r))\,dr\right)\right| \le \varphi_p^{-1}\left(q_\alpha\right)\varphi_p^{-1}\left(\int_{\delta_1}^{1-\delta_1} p(r)\,dr\right) = L.$$

Put $\delta_2 = \varepsilon / L$, then for $t_1, t_2 \in [\delta_1, 1 - \delta_1]$, $|t_1 - t_2| < \delta_2$,

$$\left|(Ty)\left(t_1\right) - (Ty)\left(t_2\right)\right| \le |(Ty)'(\xi)|\left|t_1 - t_2\right| < L\delta_2 = \varepsilon,$$

where ξ lies between t_1 and t_2. Set $\delta = \min\{\delta_1, \delta_2\}$. Then for $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta$,

$$\left|(Ty)\left(t_1\right) - (Ty)\left(t_2\right)\right| < \varepsilon, \tag{9}$$

which shows that $T\overline{B(\theta,\alpha)}$ is equicontinuous on $[0,1]$.

We next claim that $T : \overline{B(\theta,\alpha)} \to B[0,1]$ is continuous. Assume that $\{y_n\}_{n=0}^\infty \subset \overline{B(\theta,\alpha)}$ and $y_n(t)$ converges to $y_0(t)$ uniform on $[0,1]$. Hence, $\{(Ty_n)(t)\}_{n=1}^\infty$ is uniformly bounded and equicontinuous on $[0,1]$. The Arzela-Ascoli Theorem tells us that there exist uniformly convergent subsequences in $\{(Ty_n)(t)\}_{n=1}^\infty$. Let $\{(Ty_{n(m)})(t)\}_{m=1}^\infty$ be a subsequence which converges to $v(t)$ uniformly on $[0,1]$ and $\{A_{n(m)}\}_{m=1}^\infty$ converges to $\bar{A}$. In addition,

$$u_\theta(t) \le \left(Ty_n\right)(t) \le u_\alpha(t).$$

Put

$$[a, b] = \left\{t \in [0,1] : u_\alpha(t) \ge \max u_\theta(t) > 0\right\}.$$

Then $[a, b] \subset (0,1)$ and $\{A_n\} \subset [a, b]$, where A_n is the maximum point of $(Ty_n)(t)$ in $(0,1)$. Thus,

$$\begin{aligned}
\left(Ty_n\right)\left(A_n\right) &= \int_0^{A_n} \varphi_p^{-1}\left(\int_s^{A_n} f\left(r, y_n(r)\right)\,dr\right) ds \\
&\le \varphi_p^{-1}\left(q_\alpha\right)\int_0^b \varphi_p^{-1}\left(\int_s^b p(r)\,dr\right) ds,
\end{aligned}$$

$$\begin{aligned}
\left(Ty_n\right)\left(A_n\right) &= \int_{A_n}^1 \varphi_p^{-1}\left(\int_{A_n}^s f\left(r, y_n(r)\right)\,dr\right) ds \\
&\le \varphi_p^{-1}\left(q_\alpha\right)\int_a^1 \varphi_p^{-1}\left(\int_a^s p(r)\,dr\right) ds.
\end{aligned}$$

Notice that

$$(Ty_n)(t) = \begin{cases} \displaystyle\int_0^t \varphi_p^{-1}\left(\int_s^{A_n} f(r, y_n(r))\, dr\right) ds, & 0 \le t \le A_n, \\ \displaystyle\int_t^1 \varphi_p^{-1}\left(\int_{A_n}^s f(r, y_n(r))\, dr\right) ds, & A_n \le t \le 1. \end{cases}$$

Inserting $y_{n(m)}$ and $A_{n(m)}$ into the above and then letting $m \to \infty$, we obtain

$$v(t) = \begin{cases} \displaystyle\int_0^t \varphi_p^{-1}\left(\int_s^{\bar{A}} f(r, y_0(r))\, dr\right) ds, & 0 \le t \le \bar{A}, \\ \displaystyle\int_t^1 \varphi_p^{-1}\left(\int_{\bar{A}}^s f(r, y_0(r))\, dr\right) ds, & \bar{A} \le t \le 1, \end{cases} \tag{10}$$

and

$$v(\bar{A}) = \int_0^{\bar{A}} \varphi_p^{-1}\left(\int_s^{\bar{A}} f(r, y_0(r))\, dr\right) ds = \int_{\bar{A}}^1 \varphi_p^{-1}\left(\int_{\bar{A}}^s f(r, y_0(r))\, dr\right) ds. \tag{11}$$

Here we have applied Lebesgue's dominated convergence theorem. From the definition of T, we know that $v(t) \equiv (Ty_0)(t)$ on $[0,1]$. This shows that each subsequence of $\{(Ty_n)(t)\}_{n=1}^{\infty}$ uniformly converges to $(Ty_0)(t)$. Therefore, the sequence $\{(Ty_n)(t)\}_{n=1}^{\infty}$ itself uniformly converges to $(Ty_0)(t)$. This means that T is continuous at $y_0 \in \overline{B(\theta,\alpha)}$. Therefore, T is continuous on $\overline{B(\theta,\alpha)}$ since $y_0 \in \overline{B(\theta,\alpha)}$ is arbitrary. Thus, the Arzela-Ascoli Theorem implies that $T : \overline{B(\theta,\alpha)} \to B[0,1]$ is compact map. ∎

PROOF OF THEOREM 2.1. It is easily verified that $y(t)$ is a positive solution. $\Leftrightarrow$ y is a fixed point of the map T in $C^+[0,1]$,

$$(Ty)(t) := \begin{cases} \displaystyle\int_0^t \varphi_p^{-1}\left(\int_s^{A} f(r, y(r))\, dr\right) ds, & 0 \le t \le A, \\ \displaystyle\int_t^1 \varphi_p^{-1}\left(\int_{A}^s f(r, y(r))\, dr\right) ds, & A \le t \le 1. \end{cases}$$

By Lemma 2.1, $T : \overline{B}_\alpha \to B[0,1]$ is a compact map. We next claim that T satisfies Corollary 1.1.

(i) $\forall\, y \in \partial B_\alpha$, $\|y\|_0 = \alpha$. If y is convex, $g_\alpha(s) \le y(s) \le \alpha$, $s \in [0,1]$. (H2) implies that

$$\begin{aligned} \|Ty\|_0 &= \int_0^{A} \varphi_p^{-1}\left(\int_s^{A} f(r, y(r))\, dr\right) ds = \int_{A}^1 \varphi_p^{-1}\left(\int_{A}^s f(r, y(r))\, dr\right) ds \\ &\le \max\left\{\int_0^{1/2} \varphi_p^{-1}\left(\int_s^{1/2} p(r) \max_{y\in[g_\alpha(r),\alpha]} q(y)\, dr\right) ds,\right. \\ &\quad \times \left.\int_{1/2}^1 \varphi_p^{-1}\left(\int_{1/2}^s p(r) \max_{y\in[g_\alpha(r),\alpha]} q(y)\, dr\right) ds\right\} \le \alpha. \end{aligned} \tag{12}$$

(ii) $\forall\, y \in \partial B_\beta$, $\|y\|_0 = \beta$. If y is convex, $g_\beta(s) \le y(s) \le \beta$, $s \in [0,1]$. (H3) implies that

$$\begin{aligned} \|Ty\|_0 &= \int_0^{A} \varphi_p^{-1}\left(\int_s^{A} f(r, y(r))\, dr)\right) ds = \int_{A}^1 \varphi_p^{-1}\left(\int_{A}^s f(r, y(r)))\, dr)\right) ds \\ &\ge \min\left\{\int_0^{1/2} \varphi_p^{-1}\left(\int_s^{1/2} \min_{y\in[g_\beta(r),\beta]} f(r,y)\, dr\right) ds,\right. \\ &\quad \times \left.\int_{1/2}^1 \varphi_p^{-1}\left(\int_{1/2}^s \min_{y\in[g_\beta(r),\beta]} f(r,y)\, dr\right) ds\right\} \ge \beta. \end{aligned} \tag{13}$$

So, BVP (1) has at least one solution $y \in C^1[0,1], \varphi_p(y') \in C(0,1)$, and $\beta \le \|y\|_0 \le \alpha$. ∎

REMARK 2.1. The proof is not completed by using the Schauder fixed-point theorem because $f(t,y)$ is not monotone about y in $(0,\infty)$.

REMARK 2.2. For the particular case of $p=2$, the condition $0 < \int_0^{1/2} \varphi_p^{-1}(\int_s^{1/2} p(r)\,dr)\,ds + \int_{1/2}^1 \varphi_p^{-1}(\int_{1/2}^s p(r)\,dr)\,ds < \infty$ is equivalent to $t(1-t)p(t) \in L^1[0,1]$.

COROLLARY 2.1. *Suppose (H1) satisfies and $f(t,y)$ is nondecreasing about y in $(0,\infty)$. In addition, assume*

(H4) $\exists\,\alpha > \beta > 0$, *such that*

(i) $\max\{\int_0^{1/2} \varphi_p^{-1}(\int_s^{1/2} f(r,\alpha)\,dr)\,ds,\ \int_{1/2}^1 \varphi_p^{-1}(\int_{1/2}^s f(r,\alpha)\,dr)\,ds\} \le \alpha$,

(ii) $\min\{\int_0^{1/2} \varphi_p^{-1}(\int_s^{1/2} f(r,g_\beta(r))\,dr)\,ds,\ \int_{1/2}^1 \varphi_p^{-1}(\int_{1/2}^s f(r,g_\beta(r))\,dr)\,ds\} \ge \beta$,

(iii) $\exists\,t_0 \in (0,1)$, *such that* $f(t_0,0) > 0$.

Then BVP (1) has at least one solution $y \in C^1[0,1], \varphi_p(y') \in C(0,1)$, *and* $\beta \le \|y\|_0 \le \alpha$.

REMARK 2.3. For the boundary value problem

$$\begin{aligned} (\varphi_p(y'))' + f(t,y) &= 0, \qquad t \in (0,1), \\ y(0) &= 0, \qquad y'(1) = c, \end{aligned} \tag{14}$$

we have a similar result. It is omitted.

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